Probability Theory

Probability and Statistics for Data Science CSE594 - Spring 2016

What is Probability?

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Examples

- outcome of flipping a coin (seminal example)
- amount of snowfall
- mentioning a word
- mentioning a word "a lot"

What is Probability?

The chance that something will happen.

Given infinite observations of an event, the proportion of observations where a given outcome happens.

Strength of belief that something is true.

"Mathematical language for quantifying uncertainty" - Wasserman

 $\pmb{\Omega}$: Sample Space, set of all outcomes of a random experiment

A : Event ($A \subseteq \Omega$), collection of possible outcomes of an experiment

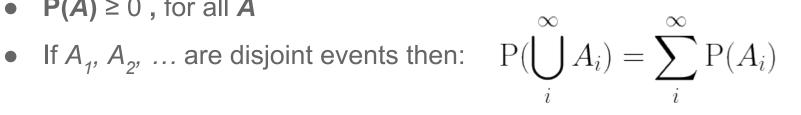
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- **P(Ω)** = 1
- $P(A) \ge 0$, for all A



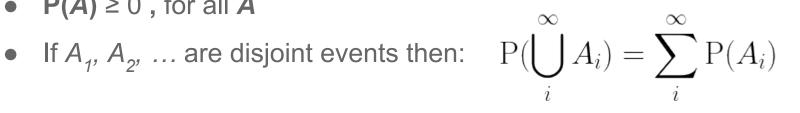
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P is a *probability measure*, if and only if

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Some Properties:

- If $B \subseteq A$ then $P(A) \ge P(B)$
- $\mathsf{P}(\mathsf{A} \cup \mathsf{B}) \leq \mathsf{P}(\mathsf{A}) + \mathsf{P}(\mathsf{B})$
- $P(A \cap B) \leq min(P(A), P(B))$
- $\mathsf{P}(\neg A) = \mathsf{P}(\Omega / A) = 1 \mathsf{P}(A)$

/ is set difference $P(A \cap B)$ will be notated as P(A, B)

Independence

Two Events: A and B

Does knowing something about A tell us whether B happens (and vice versa)?

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 B: mention or not of the word "birthday"

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Two events, A and B, are *independent* iff P(A, B) = P(A)P(B)

Conditional Probability

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P(H) = .01 P(B) = .001 P(H, B) = .0005 P(H|B) = ??

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P(H) = .01 P(B) = .001 P(H, B) = .0005P(H|B) = .50

H1: first flip of a fair coin is heads H2: second flip of the same coin is heads P(H2) = 0.5 P(H1) = 0.5 P(H2, H1) = 0.25P(H2|H1) = 0.5

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Interpretation of Independence:

Observing B has no effect on probability of A.

Why Probability?

Why Probability?

A formality to make sense of the world.

- To quantify uncertainty Should we believe something or not? Is it a meaningful difference?
- To be able to generalize from one situation or point in time to another. *Can we rely on some information? What is the chance Y happens?*
- To organize data into meaningful groups or "dimensions" Where does X belong? What words are similar to X?

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X is a *discrete random variable* if it takes only a countable number of values.

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X is a *continuous random variable* if it can take on an infinite number of values between any two given values. X is a *discrete random variable* if it takes only a countable number of values.

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What is the probability we receive (at least) a inches? $P(X \ge a) := P(\{\omega : X(\omega) \ge a\})$

What is the probability we receive between a and b inches? $P(a \le X \le b) := P(\{\omega : a \le X(\omega) \ge b\})$

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 $\mathbf{X}(\boldsymbol{\omega}) = \boldsymbol{\omega}$

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X amount of inches in a snowstorm

 $\mathbf{P}(\mathbf{X} = \mathbf{i}) := \mathbf{0}$, for all $\mathbf{i} \in \mathbf{\Omega}$

(probability of receiving <u>exactly</u> i inches of snowfall is zero)

- what constitutes a probability measure?
- independence
- conditional probability
- random variables
 - o discrete
 - \circ continuous